Introduction to MR for physicists

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Basic physics Magnetic resonance imaging (MRI) is based on nuclear magnetic resonance (NMR) which exploits how the magnetic moments of nuclei can be manipulated by magnetic fields. Firstly, nuclear magnetic spins in tissue are polarised by a static magnetic field. Secondly, oscillating magnetic field pulses at radio frequency are applied in a plane perpendicular to the static magnetic field. This perturbes the equilibrium distribution. After the pulse is turned off, the protons will relax back to thermal equilibrium, and during this process, they emit energy that can be detected by coils around the patient.

The proton has spin $S = 1/2$ and a magnetic moment of $\boldsymbol{\mu} = \gamma \mathbf{S}$, where $\mathbf{S}$ is the spin operator. The gyromagnetic constant γ, depends on the nucleus: the water proton ^{1}H, which is the main nucleus of MRI, $\gamma \simeq 2.675 \times 10^8 \mathrm{rad}\,\mathrm{T}^{-1}\,\mathrm{s}^{-1}$. In the absence of a magnetic field, the two quantum states corresponding to $m_s = \pm 1/2$ are equally populated, and the net magnetization density $\mathbf{M}$

$$\mathbf{M} = \frac{1}{V}\sum_i^N \boldsymbol{\mu}_i, \tag{1}$$

where V is volume, is 0 (its expectation value). When a magnetic field $\mathbf{B}_0$ along the z-axis is applied there is an interaction energy given by

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}_0 = -\gamma S_z B_0. \tag{2}$$

In MRI, a static magnetic field $\mathbf{B}_0$ is generated by a superconducting magnet, and for human MRI typically has values of 1.5T or 3T – up to 120,000 times the Earth's magnetic field. Thus, the two states have different energies and will be populated with different probabilities – the spin system is polarised. The corresponding equilibrium magnetization is $\mathbf{M} = M_0\hat{z}$, where

$$M_0 = \rho \frac{S(S+1)\gamma^2\hbar^2 B_0}{3k_B T} \tag{3}$$

and ρ is the spin density. The energy difference between the two states is given by

$$\Delta E = 2\mu B = \hbar\omega_0, \tag{4}$$

where $\omega_0 = \gamma B_0$ is the Larmor frequency and $\hbar$ is Planck's reduced constant. This is illustrated in Fig. 1. The energy difference is very small $(1 - \exp(-\Delta E/k_B T) \sim 10^{-6})$ and although there is a net magnetisation along the z-axis, the total magnetisation M_0 is not measured directly. If $\mathbf{M}$ acquires a component in the plane orthogonal to $\mathbf{B}_0$, also called the transverse magnetization, it will oscillate and can be measured by the current generated by induction.

Equation of Motion To see how to generate a transverse magnetization, we need to consider the dynamics of the magnetization. Classically, a magnetic dipole in a magnetic field feels a torque given by

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{\mu} \times \mathbf{B}_0 && (5) \\ &= \gamma \mathbf{S} \times \mathbf{B}_0, && (6) \end{aligned}$$

where γ is the gyromagnetic ratio and $\mathbf{S}$ is the (spin) angular momentum. Since $d\mathbf{S}/dt = \boldsymbol{\tau}$ we have

$$\frac{d\boldsymbol{\mu}}{dt} = \gamma \boldsymbol{\mu} \times \mathbf{B}_0. \quad (7)$$

Thus the change in the magnetic moment $d\boldsymbol{\mu}$ is perpendicular to $\boldsymbol{\mu}$ implying that the length of the magnetic moment is conserved, and the dynamics must be some kind of rotation. $d\boldsymbol{\mu}$ is also perpendicular to $\mathbf{B_0}$, and the resulting rotational motion of $\boldsymbol{\mu}$ around $\mathbf{B_0}$ is called *precession* and is illustrated in Fig. 1. To find the precession frequency, we consider the magnitude of the change $\left|d\boldsymbol{\mu}\right|$ during a small time interval dt

$$\begin{aligned} \left|d\boldsymbol{\mu}\right| &= \mu \sin\theta d\phi \quad \text{(from Fig. 1)} \\ &= \left|\gamma \boldsymbol{\mu} \times \mathbf{B}_0\right| = \gamma \mu B_0 \sin\theta dt, \end{aligned}$$

which (after considering the sign) means

$$\frac{d\phi}{dt} = -\gamma B_0 \equiv -\omega_0. \quad (8)$$

For a collection of non-interacting spins, the magnetization $\mathbf{M}$ obeys the same equation as $\boldsymbol{\mu}$ (Eq. 7), which holds in general for an arbitrary time-dependent magnetic field, $\mathbf{B}_0 \to \mathbf{B}(t)$. Further, since quantum mechanical expectation values obey the classical laws, we thus have

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}. \quad (9)$$

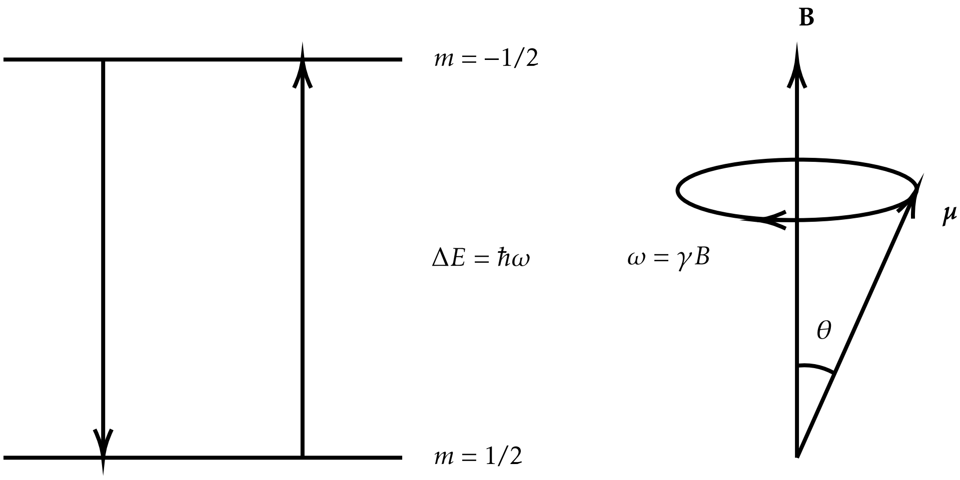

Figure 1: Left: the two states. Right: The magnetic moment, $\boldsymbol{\mu}$ precesses around the external magnetic field, $\mathbf{B}$

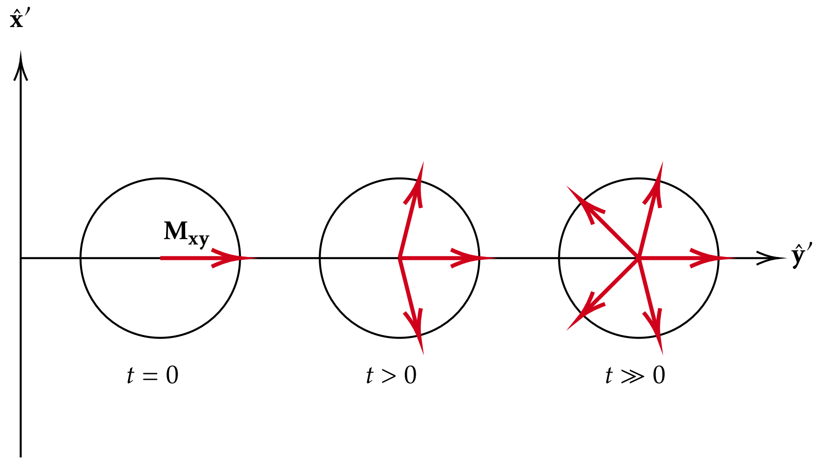

Figure 2: Dephasing of the transverse magnetization related to T_2. The red arrows are individual dipole moments μ.

The resulting motion is conveniently viewed in a frame of reference rotating at the frequency $|\Omega|$ about the z-axis, $\mathbf{\Omega} = -\Omega\hat{\mathbf{z}}$. This system is labeled by (x', y', z'). This means Eq. 9) is simply transformed into

$$\frac{d\mathbf{M}'}{dt} = \gamma \mathbf{M}' \times \mathbf{B}_{\mathrm{eff}} \tag{10}$$

where $\mathbf{B}_{\mathrm{eff}}$ is the effective magnetic field and is given by

$$\mathbf{B}_{\mathrm{eff}} = \mathbf{B} + \frac{\mathbf{\Omega}}{\gamma} \tag{11}$$

If we choose the rotating coordinate system to rotate at the Larmor frequency $\mathbf{\Omega} = -\omega_0 \hat{z}$, the spins will look stationary if the magnetic field is $\mathbf{B} = B_0 \hat{z}$, since $\mathbf{B}_{\mathrm{eff}} = 0$. If we add an RF-field oscillating at frequency ω_0 with the magnetic component $\mathbf{B_1} = b_1(\hat{x}\cos\omega_0 t - \hat{y}\sin\omega_0 t) = b_1\hat{x}'$, the effective magnetic field will be $\mathbf{B}_{\mathrm{eff}} = \mathbf{B} + \mathbf{\Omega}/\gamma = \mathbf{B}_0 + \mathbf{B_1} + \mathbf{\Omega}/\gamma = b_1\hat{x}'$. Hence,

$$\frac{d\mathbf{M}'}{dt} = \gamma \mathbf{M}' \times b_1 \hat{x}' \tag{12}$$

Therefore, in the rotating coordinate system, the dipole or the magnetization will precess around $b_1\hat{x}'$ with a frequency of $\gamma b_1 \equiv \omega_1$. This can be utilized to bring the magnetization from the equilibrium value to the transverse plane, e.g. by a so-called $\pi/2$-pulse, achieved by keeping the RF pulse on for the appropriate amount of time τ, i.e., $\tau\omega_1 = \pi/2$. Other *flip angles* $\theta \equiv \omega_1 \tau$ are also used frequently. After the RF pulse is turned off, the magnetization will continue to precess around $\mathbf{B}_0$ in the laboratory frame, and the associated time-dependent magnetic flux can be measured by the emf generated in nearby coils. This is the MR signal.
Note that if B1 is not exactly on resonance, e.g. $\mathbf{B_1} = b_1(\hat{x}\cos\omega_{\mathrm{rf}} t - \hat{y}\sin\omega_{\mathrm{rf}} t)$ with $\omega_{\mathrm{rf}} = \omega_0 - \delta\omega$, the effective magnetic field in the frame rotating with $\mathbf{B}_1$ (i.e. $\mathbf{\Omega} = -\omega_{\mathrm{rf}}\hat{z}$) is

$$\mathbf{B}_{\mathrm{eff}} = \mathbf{B} + \mathbf{\Omega}/\gamma = \mathbf{B}_0 + \mathbf{B_1} + \mathbf{\Omega}/\gamma = B_0\hat{z} + b_1\hat{x}' - (\omega_0 - \delta\omega)/\gamma\,\hat{z} = b_1\hat{x}' + \delta\omega/\gamma\,\hat{z}. \tag{13}$$

Remember we choose the rotating system precisely such that $\mathbf{B_1} = b_1\hat{x}'$ (in general some combination of $\hat{x}'$ and $\hat{y}'$ defined by a phase ϕ).

Relaxation The generated magnetization will thereafter decay towards equilibrium again via its interactions with the environment. Two processes are involved, the return of the transverse magnetization to 0 and of the longitudinal magnetization to M_0: these are termed transverse and longitudinal relaxation, respectively.

Longitudinal relaxation involves the reestablishment of M_z to the Boltzmann equilibrium distribution. This means that the proton spins will exchange energy with the surrounding "lattice" causing $\mathbf{M}$ to approach $\mathbf{M_0}$ which is parallel to $\mathbf{B}_0$. Affecting only M_z, the rate of longitudinal relaxation is phenomenologically modelled by assuming it to be proportional to the difference $M_z(t) - M_0$ with a time constant T_1. See Fig. 3a.

Transverse relaxation involves the decoherence of the individual spins driven by inhomogeneities in the field due to local atomic and nuclear effects. This means that individual protons precess at slightly different rates and the signal decays because the component of $\mathbf{M}$ orthogonal to $\mathbf{B}_0$ decreases as the individual moments loose phase coherence with an associated time constant called T_2. See Fig. 2 and Fig. 3b. We have in general $T_1 \geq T_2$, and for MRI of biological tissues, typically T_1 (hundreds of ms) is substantially larger than T_2 (tens of ms).

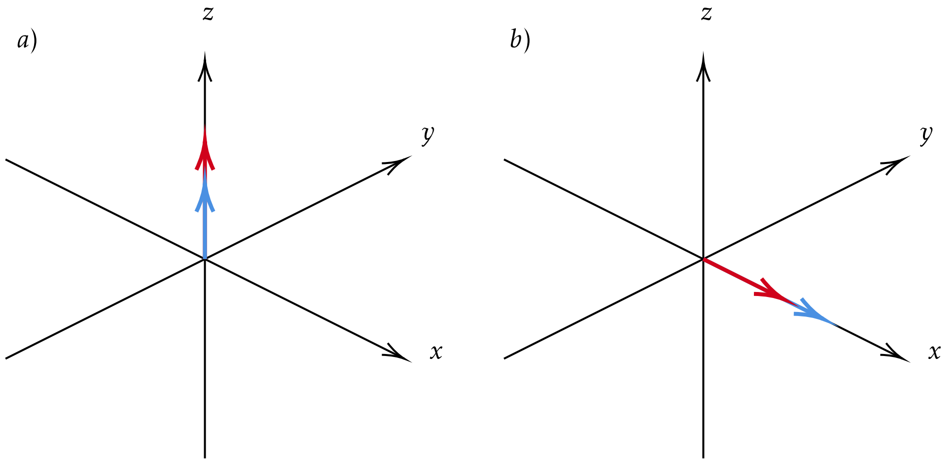

Figure 3: *a*) Longitudinal relaxation causes recovery of the longitudinal (z) component of magnetization (from blue to red) toward M_0 with an exponential time constant T_1. *b*) Transverse relaxation causes shortening of the transverse component of magnetization (from blue to red), with an exponential time-constant T_2.

Bloch Equations To more fully describe the dynamics of the magnetisation we have to incorporate the two times T_1 and T_2 into our previous equation of motion Eq. (9). This is the phenomenological Bloch equation, and is given in the rotating frame by

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}_{\text{eff}} - R_2(M_x\hat{\mathbf{x}} + M_y\hat{\mathbf{y}}) - R_1(M_z - M_0)\hat{\mathbf{z}}, \tag{14}$$

where R_2 and R_1 are given by $1/T_2$ and $1/T_1$ respectively (and we dropped the primes signifying the rotating system).

It is instructive to consider the solution in the absence of RF pulses. In the lab system we then have

$$\begin{aligned} \frac{dM_x}{dt} &= -\gamma M_y B_0 - R_2 M_x \\ \frac{dM_y}{dt} &= -\gamma M_x B_0 - R_2 M_y \\ \frac{dM_z}{dt} &= -R_1(M_z - M_0) \end{aligned} \tag{15}$$

It proves convenient to define the complex transverse magnetization given by

$$M_+ = M_x + iM_y, \tag{16}$$

in terms of which Eqs. (15) becomes two decoupled equations

$$\frac{dM_+}{dt} = -i\omega_0 M_+ - R_2 M_+ \tag{17}$$
$$\frac{dM_z}{dt} = -R_1(M_z - M_0) \tag{18}$$

with the solution

$$M_+(t) = M_+(0)e^{-t/T_2}e^{-i\omega_0 t} \tag{19}$$
$$M_z(t) = M_z(0)e^{-t/T_1} + (M_z - M_0)e^{-t/T_1} \tag{20}$$

For a short $\pi/2$ at $t = 0$, $M_+(0) = iM_0$.

Spatial Encoding Magnetic field gradients can also be applied in addition to the static magnetic field to encode the signal spatially and be processed to generate images, e.g. corresponding to slices through the tissue. The gradients are generated by three orthogonal gradient coils in the scanner and produce a linearly varying magnetic field, G in the (x, y, z) direction

$$\mathbf{G} = \nabla B_z. \tag{21}$$

Magnetic fields are additive which means the total magnetic field experienced by the protons is given by

$$B_z(\mathbf{r}) = B_0 + \mathbf{G} \cdot \mathbf{r}. \tag{22}$$

Combining this with the Larmor frequency relation $\omega = \gamma B$ yields

$$\omega(\mathbf{r}) = \omega_0 + \gamma \mathbf{G} \cdot \mathbf{r}. \tag{23}$$

For example, with $\mathbf{G}$ along $\hat{z}$, this means when an RF pulse $\mathbf{B}_{\mathrm{rf}}(t)$ is applied with a narrow band Δf of frequencies around the Larmor frequency, the only protons to be resonantly excited will be those within a narrow slice of thickness $\Delta z \simeq 2\pi\Delta f/\gamma G_z$. Spins further off resonance will not be affected to any substantial degree.

To go from measuring signals to an actual image we have to introduce k-space and the Fourier transformation. To make things simpler assume 1D ($\mathbf{G} \to G_x$, say), as the results can be generalized to three dimensions. Neglecting relaxation, the MR signal is proportional to the spatial integral over the transverse magnetization in Eq. 16,

$$S(t) = \int dx\, M_+(x, t) = \int dx\, e^{-i\omega(x)t} M_+(x, 0), \tag{24}$$

where we assumed $t \ll T_2$ and used Eq. (19) for general $\mathbf{B}$ along $\hat{z}$. Writing $\omega(x)$ as

$$\omega(x) = \omega_0 + \gamma G_x x \tag{25}$$

this yields (after demodulation, i.e. removing $e^{-i\omega_0 t}$)

$$S(t) = \int dx\, e^{-iG_x\gamma t} M_+(x, 0) \tag{26}$$

or

$$S(k) = \int dx\, \mathrm{e}^{-ikr} M_+(x, 0) \tag{27}$$

where $k = G\gamma t$. This means the signal is the Fourier transform of $M_+(\mathbf{r}, 0)$! Using the inverse Fourier transform

$$M_+(x,0) = \int \frac{dk}{2\pi} \, \mathrm{e}^{-ikx} S(k). \tag{28}$$

we have our MRI image, typically depicted as $|M_+(x,0)$. Note that it requires us to sample the signal for multiple k-values, which here happens automatically as time proceeds. If we precede the application of gradients by a $\pi/2$ pulse, we can make $M_+(\mathbf{r},0) \propto \rho$. If we wait some time, we can also impart dependence on the relaxation times T_1 and T_2. Many other types of contrast can be generated by careful design of the pulse sequence.

Discrete sampling In practice we sample the signal at discrete time points over a finite time iterval. For example, N points with discrete sampling frequency $1/\Delta t$ results in measurement only at discrete wave-vectors $k_m = m\gamma G_x \Delta t = m\Delta k$. So a discrete inverse Fourier transform is applied

$$M_+(x) \simeq \frac{1}{N} \sum_m \mathrm{e}^{ik_m x} S(k_m). \tag{29}$$

The resulting function is periodic with period $2\pi/\Delta k \equiv L$, and L is called field-of-view (FOV). Due to maximal resolution in reconstructed M_+ (due to maximal k), and the need to render on a computer, we only reconstruct $M_+(x)$ at discrete x, typically $x_n = n\Delta x = nL/N$. Note that $\Delta x \Delta k = 2\pi/N$.